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Critical behaviour of Dyson's hierarchical model with a random field

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Abstract. The critical behaviour of Dyson's hierarchical Ising model is studied in the presence of a random field. Simple scaling relations determine the exponents η, $\bar{\eta}$ exactly for random fields with either short- or long-range correlations. The Schwartz-Soffer inequality $\bar{\eta} \leq 2\eta$ is satisfied as an equality provided the random-field correlations are not too long ranged. For short-range random fields, the exponent ν is calculated to order $\varepsilon = \lambda - \frac{4}{3}$, where λ is the 'range parameter' of the Dyson model. The exponent ν is also computed to $O(\varepsilon^2)$, with $\varepsilon = 2\lambda - 3$, for the hierarchical $O(n)$ model in zero field: the results reveal striking similarities with the exponents of the corresponding one-dimensional model with long-range interactions.

1. Introduction

The Dyson hierarchical model [1, 2] was introduced to clarify the existence of phase transitions in one-dimensional spin systems with long-range interactions. While simulating a system with power-law interactions, it has the advantage that renormalisation group (RG) transformations can be applied to it exactly [3]. The critical behaviour of the model is determined from the fixed-point solutions of a non-linear integral equation [4] via a linear stability analysis. The integral equation has a Gaussian solution, valid in the classical regime, for the spin weight, which can be expanded about to give the critical exponent ν in the non-classical regime. This is similar to expansions about an upper critical dimension in RG calculations for short-range systems: the 'range parameter' λ of the Dyson model plays a role similar to that played by the spatial dimensionality d in systems with short-ranged interactions.

In § 3 this calculation is performed for the $O(n)$ model in zero field, and the results compared with those given in the literature for the special case $n = 1$ (i.e. the Ising model). We find agreement with Kim and Thompson [5] but not with Blekher and Sinai [4]. We also compare our results with those for the corresponding one-dimensional $O(n)$ model [6]. To order ε^2 the only difference between the exponents ν is a small change in the numerical prefactor of the ε^2 term—there is no change in the functions of n which appear to this order!

With the close agreement between the long-range one-dimensional and hierarchical models in mind, we consider in § 2 the Dyson model with a random field. Both correlated and uncorrelated random fields are considered. The random-field correlations are chosen in a similar way to the exchange interactions—they are long range with a power-law dependence on 'distance'. A non-linear integral equation is derived and we look at the fixed points to determine perturbatively the critical exponent ν of the model. By contrast, the exponents η, $\bar{\eta}$ are determined exactly by simple scaling equations, where the observation [7-9] that the critical behaviour is controlled by a zero-temperature fixed point plays a crucial role. Two universality classes are identified—one controlled by a fixed point with a random field of short-range character, the other with one of long-range character. These results are in agreement with previous work [7].

2. The Dyson model with a random field

Consider a Dyson hierarchical model in one dimension, with coupling between spins as in figure 1. The zero-field Hamiltonian is

$$\begin{aligned} H_0 = -\sum_{i<j} J_{ij}S_iS_j \\ = -J(S_1S_2+S_3S_4+\cdots) \\ -(J/2^\lambda)[(S_1+S_2)(S_3+S_4)+(S_5+S_6)(S_7+S_8)+\ldots] \\ -(J/2^{2\lambda})[(S_1+S_2+S_3+S_4)(S_5+S_6+S_7+S_8)+\ldots]-\ldots. \end{aligned} \quad (1)$$

The first-level spin coupling J is a positive constant, and the 'range parameter' λ controls the rate of decay of the exchange interactions with increasing level in the hierarchy. The model simulates a system with long-range ferromagnetic interactions decaying with distance as $1/r^\lambda$. The partition function is given by

$$Z=\int \left(\prod_i \mathrm{d}S_i\right) \exp\left(-\sum_i W(S_i)-\beta H\right) \quad (2)$$

where β is the reciprocal of the temperature T, H is the Hamiltonian and $W(S) = W(-S)$ is the spin weight function.

If a random field h_i is added at site i, and a uniform field h is also applied, the full Hamiltonian H is

$$H = H_0 - \sum_i h_iS_i - h\sum_i S_i. \quad (3)$$

The random fields are chosen to have a Gaussian distribution, with correlations given by

$$\langle h_i\rangle_h = 0 \qquad \langle h_i^2\rangle_h = h_{\mathrm{SR}}^2 \quad (4)$$

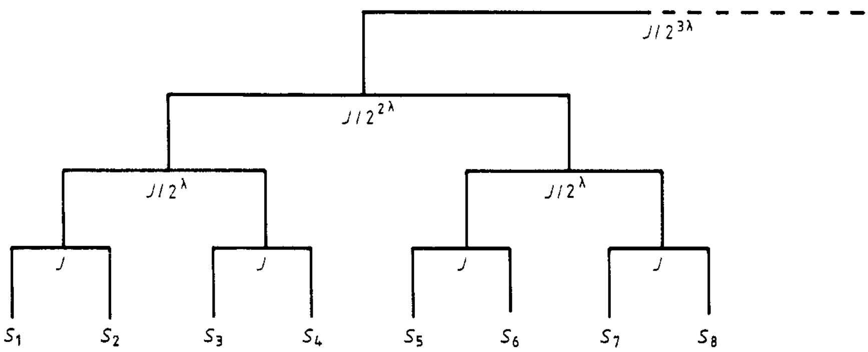

Figure 1. Dyson's hierarchical model.

and

$$\langle h_1 h_2\rangle_h = h_{\mathrm{LR}}^2 \qquad \langle h_1 h_3\rangle_h = \langle h_1 h_4\rangle_h = h_{\mathrm{LR}}^2/2^{\mu} \qquad \langle h_1 h_5\rangle_h = h_{\mathrm{LR}}^2/2^{2\mu} \tag{5}$$

and so on, where $\langle\ \rangle_h$ indicates an average with respect to the random-field distribution. The long-range random-field correlations are seen to have the same structure as the exchange couplings $\{J_{ij}\}$—they simulate a system with random-field correlations $\langle h_i h_j\rangle_h$ decaying as $1/r_{ij}^{\mu}$.

In order to average over the disorder introduced by the random fields, the replica method [10] is used. The replica index is $\alpha = 1, \ldots, n$, with $n \to 0$ ultimately. Thus

$$Z^n = \int \mathrm{D}S \exp\Big(-\sum_i W(\{S_i^\alpha\}) + \beta \sum_{(i,j)\alpha} J_{ij} S_i^\alpha S_j^\alpha + \beta \sum_i (h_i + h) \sum_\alpha S_i^\alpha\Big) \tag{6}$$

where $\mathrm{D}S \equiv \Pi_{i,\alpha}\, \mathrm{d}S_i^\alpha$. The general form of the spin weight function used in (6) allows for the possibility that different replicas will become coupled after coarse graining. Initially, $W(\{S^\alpha\}) = \Sigma_\alpha W(S^\alpha)$, with the property that W is invariant under $S^\alpha \to -S^\alpha$ for any α. We shall see that this symmetry is preserved by the coarse-graining procedure.

For Gaussian random fields

$$\Big\langle \exp\Big(\beta \sum_{i,\alpha} h_i S_i^\alpha\Big)\Big\rangle_h = \exp\Big[\tfrac{1}{2}\Big\langle\Big(\beta \sum_{i,\alpha} h_i S_i^\alpha\Big)^2\Big\rangle_h\Big] \tag{7}$$

so the field average of Z^n becomes, using (4) and (5),

$$\begin{aligned}\langle Z^n\rangle_h = \int \mathrm{D}S \exp\Big(&-\sum_i W(\{S_i^\alpha\}) + \beta \sum_{(i,j)\alpha} J_{ij} S_i^\alpha S_j^\alpha \\ &+ \tfrac{1}{2}\beta^2 h_{\mathrm{SR}}^2 \sum_{i,\alpha,\beta} S_i^\alpha S_i^\beta + \beta^2 h_{\mathrm{LR}}^2 \sum_{\alpha,\beta} (S_1^\alpha S_2^\beta + S_3^\alpha S_4^\beta + \ldots) \\ &+ (\beta^2 h_{\mathrm{LR}}^2/2^{\mu}) \sum_{\alpha,\beta} [(S_1^\alpha + S_2^\alpha)(S_3^\beta + S_4^\beta) + \ldots] + \ldots + \beta h \sum_{i,\alpha} S_i^\alpha\Big).\end{aligned} \tag{8}$$

A coarse-graining procedure can now be performed to obtain RG equations for $K \equiv \beta J$, h_{SR}, h_{LR}, h and $W(\{S_i^\alpha\})$. These are obtained in the usual way by an elimination of 'hard' modes, accompanied by a rescaling of the spin variables. The first step is to introduce 'soft' and 'hard' block-spin variables via the transformation

$$S_i^\alpha + S_{i+1}^\alpha = 2\gamma\sigma_{(i+1)/2}^\alpha \tag{9}$$

$$S_i^\alpha - S_{i+1}^\alpha = 2\tau_{(i+1)/2}^\alpha \tag{10}$$

where i takes odd values only and γ is the spin rescaling factor. Substituting (9) and (10) into (8) and integrating out the 'hard' modes $\{\tau_i^\alpha\}$ yields an expression for $\langle Z^n\rangle_h$ which, apart from a multiplicative constant, has the same form as (8) but with new coupling and field strengths,

$$K' = 2^{2-\lambda}\gamma^2 K \tag{11}$$

$$h_{\mathrm{LR}}' = 2^{1-\mu/2}\gamma h_{\mathrm{LR}} \tag{12}$$

$$(h_{\mathrm{SR}}')^2 = 2\gamma^2(h_{\mathrm{SR}}^2 + h_{\mathrm{LR}}^2) \tag{13}$$

$$h' = 2\gamma h \tag{14}$$

with

$$K \equiv \beta J \tag{15}$$

and a new spin weight $W'(\{\sigma^\alpha\})$ given by

$$\exp(-W'(\{\sigma^\alpha\})) = I(\{\sigma^\alpha\})/I(0) \tag{16}$$

where

$$I(\{\sigma^\alpha\}) = \int \left(\prod_\alpha \mathrm{d}\tau^\alpha\right) \exp\left(-W(\{\gamma\sigma^\alpha + \tau^\alpha\}) - W(\{\gamma\sigma^\alpha - \tau^\alpha\}) + \beta^2(h_{\mathrm{SR}}^2 - h_{\mathrm{LR}}^2)\sum_{\alpha\beta} \tau^\alpha\tau^\beta - K\sum_\alpha (\tau^\alpha)^2 + \gamma^2 K \sum_\alpha (\sigma^\alpha)^2\right). \tag{17}$$

Notice that the site indices have been dropped from (16) and that $W'(\sigma)$ has been normalised such that $W'(0) = 0$.

In order that (13) be exact, it is important that no terms of the form $\Sigma_{\alpha\beta}\,\sigma^\alpha\sigma^\beta$ appear in $I(\{\sigma^\alpha\})$, since these would represent additional contributions to h'_{SR}. The absence of such terms is, however, guaranteed by the symmetry of $W(\{\sigma^\alpha\})$ under $\sigma^\alpha \to -\sigma^\alpha$ for any α. According to (17), $I(\{\sigma^\alpha\})$ possesses the same symmetry, so while terms like $\Sigma_{\alpha\beta}\,(\sigma_\alpha)^2(\sigma_\beta)^2$ can be generated, terms like $\Sigma_{\alpha\beta}\,\sigma_\alpha\sigma_\beta$ cannot.

We now have all the information needed to study the critical behaviour of the hierarchical model in a random field. First we will look at the short-range problem.

2.1. Short-range random field

In order to calculate the 'upper critical parameter' λ_{u} for the short-range random-field problem (such that no phase transition is possible for $\lambda > \lambda_{\mathrm{u}}$), we note that at the zero-temperature zero-random-field fixed point (see figure 2) the spin weight is given by the bimodal distribution, $\exp\{-W(S)\} = \frac{1}{2}[\delta(S-1) + \delta(S+1)]$, with spin rescaling factor $\gamma = 1$. From (11) and (13) we get (with $h_{\mathrm{LR}} = 0$ and $\gamma = 1$)

$$(h_{\mathrm{SR}}/J)' = 2^{\lambda - 3/2}(h_{\mathrm{SR}}/J) \tag{18}$$

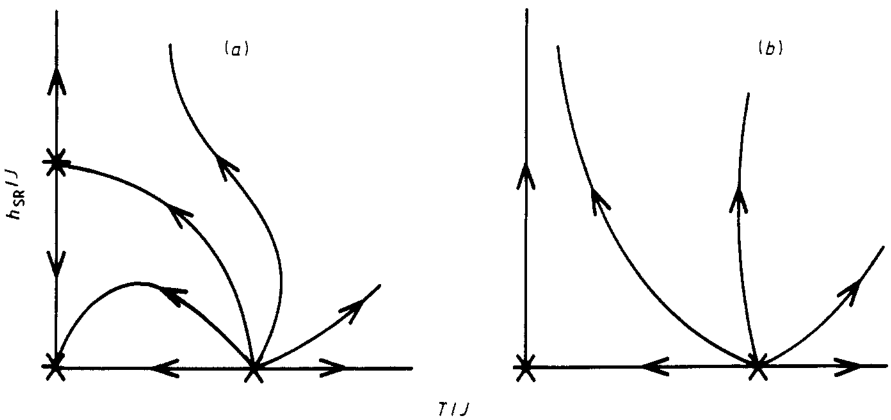

Figure 2. Schematic RG flows for the Dyson model in a random field, for the case where the random-field correlations are short-ranged. (a) $\lambda < \frac{3}{2}$, (b) $\lambda > \frac{3}{2}$.

implying $\lambda_u = \frac{3}{2}$. For $\lambda > \frac{3}{2}$, h_{SR} is a relevant perturbation at $T=0$: the ferromagnetic order is destroyed by a weak random field. For $\lambda < \frac{3}{2}$, h_{SR} is irrelevant: the zero-temperature zero-random-field fixed point is stable with respect to the random field, implying that the ferromagnetic order survives in the presence of a weak random field. Schematic RG flow diagrams are shown in figure 2.

In order to study the non-trivial zero-temperature fixed point at non-zero random field, the spin rescaling factor γ must be chosen to keep h_{SR}/J fixed. From (11) and (13) this requires $\gamma = 2^{\lambda-3/2}$ at the non-trivial fixed point, giving

$$h' = 2^{\lambda-1/2}h \tag{19}$$

$$J' = 2^{\lambda-1}J. \tag{20}$$

Scaling laws derived by Bray and Moore [9] may now be used to compute the critical exponents η and $\bar{\eta}$ which describe the decay at criticality of the connected and disconnected correlation functions respectively [9]:

$$\langle\langle S_0 S_r\rangle_T - \langle S_0\rangle_T\langle S_r\rangle_T\rangle_h \sim T/r^{d-2+\eta} \tag{21}$$

$$\langle\langle S_0\rangle_T\langle S_r\rangle_T\rangle_h \sim 1/r^{d-4+\bar{\eta}} \tag{22}$$

where $\langle\ \rangle_T$ indicates a thermal average. On the Dyson lattice $d=1$, and r has to be reinterpreted as 2^l, where l is the level of the hierarchy through which sites 0 and r are connected. Bray and Moore [9] found that if, after coarse graining at the non-trivial zero-temperature fixed point, the rescaling of h and J is given by

$$h' = b^x h \qquad J' = b^y J \tag{23}$$

where b is the length rescaling factor, then

$$\eta = d+2+y-2x \qquad \bar{\eta} = d+4-2x. \tag{24}$$

Using (19), (20) and $d=1$ yields

$$\bar{\eta} = 6-2\lambda = 2\eta. \tag{25}$$

The exponents η, $\bar{\eta}$ satisfy the Schwartz–Soffer inequality [11] $\bar{\eta} \leq 2\eta$ as an equality. Thus two of the three independent exponents [8, 9] are trivially determined on the Dyson lattice, for all $\lambda < \lambda_u$.

It is also possible to calculate the correlation length exponent ν for λ close to the lower critical value λ_l such that 'classical' exponents are obtained for $\lambda < \lambda_l$. Putting $W(\{\sigma^\alpha\}) = r\sum_\alpha (\sigma^\alpha)^2 + u\sum_\alpha (\sigma^\alpha)^4$ into (17), and treating the σ^4 term perturbatively, gives, to lowerst non-trivial order in uh_{SR}^2,

$$r' = \gamma^2(2r - K + 6u\beta^2 h_{SR}^2 q^2) \tag{26}$$

$$u' = \gamma^4(2u - 72u^2\beta^2 h_{SR}^2 q^3) \tag{27}$$

where $q = 1/(K+2r)$ and, as noted earlier, $\gamma = 2^{\lambda-3/2}$. Terms in u in (26), and in u^2 in (27) (without accompanying factors of h_{SR}^2), have been dropped, as these correspond to irrelevant thermal fluctuations, and the limit $n \to 0$ has been taken explicitly. Making the changes of variable $\tilde{r} = r/K$, $\tilde{q} = qK$, $\tilde{u} = (\beta^2 h_{SR}^2/K^3)u$ yields

$$\tilde{r}' = 2^{\lambda-1}(\tilde{r} - \tfrac{1}{2} + 3\tilde{u}\tilde{q}^2) \tag{28}$$

$$\tilde{u}' = 2^{3\lambda-4}(\tilde{u} - 36\tilde{u}^2\tilde{q}^3). \tag{29}$$

Thus $\lambda_l = \frac{4}{3}$: for $\lambda < \frac{4}{3}$, $\tilde{u}$ is an irrelevant perturbation at the Gaussian fixed point and $1/\nu = \lambda - 1$. For $\lambda > \frac{4}{3}$, with $\varepsilon = \lambda - \frac{4}{3}$ small, linearising (28) and (29) about the non-Gaussian fixed point in the standard way gives

$$1/\nu = \lambda - 1 - \varepsilon + O(\varepsilon^2). \tag{30}$$

As mentioned above, the coarse-graining procedure eventually couples different replicas together, so it might be wondered if the assumed form for $W(\{\sigma^\alpha\})$ is adequate. The leading term coupling different replicas, and consistent with the symmetry of the problem, has the form $v\,\Sigma_{\alpha\beta}\,(\sigma_\alpha)^2(\sigma_\beta)^2$. It is easy to show that the fixed-point value of v is order ε^2, so that (30) is unchanged at order ε.

2.2. *Long-range random field*

To determine whether a long-range random field is a relevant perturbation at the short-range random-field fixed point, we look at the scaling of $x \equiv (h_{LR}/h_{SR})^2$ under the RG. From (12) and (13) we find

$$x' = 2^{1-\mu} x/(1+x) \tag{31}$$

so a long-range random field is relevant if (and only if) $\mu < 1$. In the following we restrict ourselves to this case.

To determine the upper critical parameter λ_u we follow the method used in § 2.1 and look at the zero-temperature zero-random-field fixed point. From (11) and (12), with $\gamma = 1$,

$$(h_{LR}/J)' = 2^{\lambda-1-\mu/2}(h_{LR}/J). \tag{32}$$

Hence $\lambda_u = 1 + \frac{1}{2}\mu$: for $\lambda < 1 + \frac{1}{2}\mu$, h_{LR} is an irrelevant perturbation (with respect to the exchange coupling) and the ordered phase is stable against a weak random field.

At the non-trivial zero-temperature fixed point, $\gamma = 2^{\lambda-1-\mu/2}$ keeps h_{LR}/J fixed and (11) and (14) yield

$$h' = 2^{\lambda-\mu/2} h \tag{33}$$

$$J' = 2^{\lambda-\mu} J. \tag{34}$$

Using (23) and (24), with $d = 1$, gives

$$\eta = 3 - \lambda \qquad \bar{\eta} = 5 - 2\lambda + \mu. \tag{35}$$

Note that $2\eta - \bar{\eta} = 1 - \mu > 0$ at the fixed point controlled by the long-range random field, so the Schwartz–Soffer inequality [11], $2\eta - \bar{\eta} \geqslant 0$, is again satisfied.

It is clear that there are two universality classes for the Dyson model with random fields. One is associated with random fields of short-range character, or long-range with $\mu > 1$, the other with long-range random fields with $\mu < 1$. For the former class, $2\eta - \bar{\eta} = 0$; for the latter, $2\eta - \bar{\eta} = 1 - \mu$. In all cases $\eta = 3 - \lambda$. These two universality classes correspond to two (called LRE and LREF in [7]) of the four identified by one of us [7] for general systems with long-range exchange interactions and/or long-range random fields.

3. O(n) ferromagnet in zero field

In this section we demonstrate the very close agreement between the critical exponents for the Dyson model of a ferromagnet and the exponents of the corresponding one-dimensional model with power-law interactions. For generality we consider an O(n) model, i.e. a ferromagnet in which the spins are vectors in an n-dimensional space. The comparison we make is between the Dyson model with parameter λ and the one-dimensional model with interactions falling off as $r^{-\lambda}$ ($\lambda < 2$ is assumed, to ensure a phase transition [1, 2]). The exponent η is known to be the same for both models: $\eta = 3 - \lambda$. Below, we show that the exponent ν is also remarkably close for the two models, at least to order ε^2, where $\varepsilon = 2\lambda - 3$ measures the deviation of λ from its lower critical value $\lambda_l = \frac{3}{2}$. This leads us to believe that results obtained via the Dyson model should be reasonably reliable, at least qualitatively (see, however, the discussion in the following section).

The partition function for the Dyson O(n) model is

$$Z = \int \left(\prod_i \mathrm{d}^n S_i\right) \exp\left(-\sum_i W(\boldsymbol{S}_i) + \beta \sum_{(i,j)} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j\right). \tag{36}$$

Coarse graining and rescaling the spin variables as before, via the transformations

$$\boldsymbol{S}_i + \boldsymbol{S}_{i+1} = 2\gamma \boldsymbol{\sigma}_{(i+1)/2} \tag{37}$$

$$\boldsymbol{S}_i - \boldsymbol{S}_{i+1} = 2\boldsymbol{\tau}_{(i+1)/2} \tag{38}$$

for i odd, and integrating out the hard modes $\{\boldsymbol{\tau}\}$, we obtain

$$K' = (4\gamma^2/2^\lambda)K \tag{39}$$

$$\exp[-W'(\sigma)] = I(\sigma)/I(0) \tag{40}$$

where

$$I(\sigma) = \int \mathrm{d}^n \tau \exp[-W(\gamma\boldsymbol{\sigma} + \boldsymbol{\tau}) - W(\gamma\boldsymbol{\sigma} - \boldsymbol{\tau}) + K(\gamma^2 \boldsymbol{\sigma}^2 - \boldsymbol{\tau}^2)]. \tag{41}$$

Equations (39)-(41) have a Gaussian fixed point at non-zero temperature. At such a fixed point K is constant, so from (39) $\gamma = 2^{\lambda/2-1}$. For $\lambda < \frac{3}{2}$, the Gaussian fixed point is stable, and yields [4, 5] $1/\nu = \lambda - 1$. For $\varepsilon = 2\lambda - 3$ small and positive, ν can be computed as a power series in ε by including perturbative corrections to the Gaussian spin weight. This process is simplified by the following change of variables:

$$W(\boldsymbol{S}) = P(\boldsymbol{S}) + \frac{2^{\lambda-2}K}{2^{\lambda-1} - 1} \boldsymbol{S}^2. \tag{42}$$

Note that neither the critical (i.e. fixed-point) value of K nor the fixed-point spin weight are universal, but depend on the choice of initial spin weight. The value of K at the fixed point is a free parameter—different choices lead to different fixed-point spin weights, but the same set of critical exponents. A convenient choice is $K = (2^{\lambda-1} - 1)/(2^\lambda - 1)$, which yields

$$\exp[-P'(\boldsymbol{\sigma})] = J(\boldsymbol{\sigma})/J(0) \tag{43}$$

with

$$J(\boldsymbol{\sigma}) = \int \mathrm{d}^n \tau \exp[-P(\gamma\boldsymbol{\sigma} + \boldsymbol{\tau}) - P(\gamma\boldsymbol{\sigma} - \boldsymbol{\tau}) - \boldsymbol{\tau}^2]. \tag{44}$$

This equation is identical to that derived by Baker [3] for the hierarchical model. It is the same as the approximate RG equation of Wilson [12], in one dimension for $\lambda = 3$.

To expand about the Gaussian fixed point we put

$$P(\boldsymbol{\sigma}) = r\boldsymbol{\sigma}^2 + u\boldsymbol{\sigma}^4 + v\boldsymbol{\sigma}^6 \tag{45}$$

in (44). Retaining terms up to $O(\boldsymbol{\sigma}^6)$ is sufficient to compute ν to $O(\varepsilon^2)$. Expanding the exponential in (44) in u and v yields recursion relations

$$r' = 2^{\lambda-1}[r + (n+2)uq - 2(n+2)^2u^2q^3 + \ldots] \tag{46}$$

$$u' = 2^{2\lambda-3}[u - 2(n+8)u^2q^2 + 4(3n^2+26n+52)u^3q^4 + \tfrac{1}{2}(3n+12)vq + \ldots] \tag{47}$$

$$v' = 2^{3\lambda-5}[v + \tfrac{16}{3}(n+26)u^3q^3 + \ldots] \tag{48}$$

where $q = 1/(1+2r)$.

From (47) we see that the lower critical parameter $\lambda_l = \frac{3}{2}$. For $\lambda < \frac{3}{2}$ the Gaussian fixed point ($u = 0 = v$) is stable and, from (46), $1/\nu = \lambda - 1$. For $\varepsilon = 2\lambda - 3$ small we get, by linearising around the non-trivial fixed point of (46)-(48),

$$1/\nu = \lambda - 1 - \varepsilon\frac{(n+2)}{(n+8)} - \varepsilon^2(\ln 2)(3+4\sqrt{2})\frac{(n+2)(7n+20)}{(n+8)^3} + O(\varepsilon^3). \tag{49}$$

For $n = 1$ this result agrees with that of [5], but disagrees with that of [4], which we conclude is incorrect, at $O(\varepsilon^2)$.

Equation (49) can also be compared with the result for the one-dimensional $O(n)$ model [6] with long-range interactions decaying as $r^{-\lambda}$. Remarkably, the latter expression for $1/\nu$ has the same combinatoric factors (i.e. the same functions of n) as (49). The only difference to $O(\varepsilon^2)$ is in the overall coefficient of the ε^2 term. Instead of the factor $(\ln 2)(3+4\sqrt{2})$ the authors of [6] obtain $(4\ln 2 + \pi)$. These two numbers differ by about $1\frac{1}{2}\%$. In addition, the exponent η is the same for both models, $\eta = 3 - \lambda$.

4. Discussion

Close agreement between the ε expansions for the Dyson model and the equivalent one-dimensional model was noted in the previous section. Most remarkable is that the functions of n which appear are the same for both models. This is especially surprising if one thinks in terms of a graphical analysis of the perturbation expansions. The familiar 'water melon' diagram, for example, containing three internal lines, is absent for the Dyson model since, when (44) is expanded as a power series in $\boldsymbol{\sigma}^2$, only terms involving even powers of both σ and τ appear. The water melon diagram, which enters the RG equation for r at order u^2, contains two vertices, each coupling one 'soft' mode to three 'hard' modes. Such a vertex, corresponding to a term of the form $\sigma\tau^3$, is absent for the Dyson model. Similarly there are terms which appear in the RG equations for u and v in the one-dimensional problem which have no analogue in the Dyson model. Despite these differences, the functions of n entering the exponents, although seeming to depend crucially on the particular diagrams which appear, are in fact the same for both models, at least to order ε^2. We conclude that the Dyson model captures even more of the physics of one-dimensional systems with long-range interactions than is at first apparent.

The above considerations provided some of the motivation for studying the Dyson model with random fields. Recently, Weir *et al* [13] have studied the related one-dimensional long-range model with random fields, and it is interesting to compare their results with ours. Their approach is to derive renormalisation group equations for the Hamiltonian parameters by eliminating close pairs of domain walls. This is a systematic procedure, in principle, for λ less than, but close to, $\lambda_u = \frac{3}{2}$, with $\varepsilon \equiv \frac{3}{2} - \lambda$ appearing as a small parameter. With the assumption that the random-field distribution stays Gaussian under renormalisation, Weir *et al* obtain RG equations for J, h_{SR}, h and the chemical potential μ of the domain walls. These equations yield results identical to ours to first order in ε. Further study, however, reveals that the random-field distribution does *not*, in fact, stay Gaussian: the high-order cumulants become large after coarse graining, indicating, Weir *et al* speculate, a first-order phase transition. Further work, however, is needed to substantiate this conjecture. In the Dyson model, by contrast, the field distribution *does* stay Gaussian, as we have seen.

In conclusion, the Dyson model without random fields captures rather well the physics of the corresponding one-dimensional system with long-range interactions, whose behaviour in turn mirrors, as a function of λ, the behaviour of short-range systems as a function of d. In the presence of random fields, the Dyson model has a continuous phase transition with exponents η, $\bar{\eta}$ which satisfy the Schwartz-Soffer inequality $\bar{\eta} \leq 2\eta$ as an equality, provided the random-field correlations are not too long ranged (i.e. provided $\mu > 1$). With random fields, however, the behaviour of the Dyson model may be qualitatively different from that of the corresponding one-dimensional model [13]. Further work is needed to clarify these differences and the relation to short-range systems in higher dimensions.

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References

[1] Dyson F J 1969 *Commun. Math. Phys.* **12** 91
[2] Dyson F J 1971 *Commun. Math. Phys.* **21** 269
[3] Baker G A Jr 1972 *Phys. Rev.* B **5** 2622
[4] Blekher P M and Sinai Ya G 1974 *Sov. Phys.-JETP* **40** 195
[5] Kim D and Thompson C J 1977 *J. Phys. A: Math. Gen.* **10** 1579
[6] Fisher M E, Ma S-K and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917
[7] Bray A J 1986 *J. Phys. C: Solid State Phys.* **19** 6225
[8] Fisher D S 1985 *Phys. Rev. Lett.* **56** 416
Villain J 1985 *J. Physique* **46** 1843
Grinstein G 1976 *Phys. Rev. Lett.* **37** 944
[9] Bray A J and Moore M A 1985 *J. Phys. C: Solid State Phys.* **18** L927
[10] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[11] Schwartz M and Soffer A 1985 *Phys. Rev. Lett.* **55** 2499
[12] Wilson K G 1971 *Phys. Rev.* B **4** 3184
[13] Weir P O, Read N and Kosterlitz J M 1987 *Preprint* Brown University